downstream crossflow vortices are highly three-dimensional, and the use of simple vortex models commonly applied to low-speed jets in crossflow is questionable. Furthermore, because of the upstream constraint of the vortices and the advection of jet fluid caused by the relatively large transverse velocities above the jet plume, the location of maximum upwash velocity is not necessarily expected to occur directly between the vortex centers.

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Stiffness Matrix Adjustment Using Incomplete Measured Modes

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Introduction

THE analytically evaluated dynamic characteristics of a structure seldom agree with the corresponding measured experimental ones. The problem of the correlation between the analytical and measured modal parameters has been addressed for many years, and a number of approaches and discussions have been published on the subject.

In the line of the Baruch/Berman (see Refs. 1–4) and Kabe[5,6] methods, this Technical Note describes a new algorithm for stiffness adjustment. It is shown that the Baruch/Berman method can be considered as a general expression of the adjusted stiffness matrix. The structural connectivity constraints are applied to the generalized Baruch/Berman equation as constraints in the optimal process so that the dimension of the derived governing equation is equal to the number of zero elements in the upper-side triangular part of the stiffness matrix. Generally, the stiffness matrix is obtained from Guyan reduction and is full. Only a few elements with very small values are considered to be zero. In this case, the dimension of the derived governing equation will be relativaly small.

The derived auxiliary equation system is solved by the Householder QR decomposition, which is simple and efficient. A numerical example is presented to illustrate the performance of the method and compare the results with those obtained by the Kabe[5,6] and the Kammer[7] methods.

Method

General Expression for the Adjusted Stiffness Matrix

Given a symmetric positive definite mass matrix $M(n \times n)$ and a rectangular measured modal matrix $\phi_m(n \times m)$, $m \ll n$, which is normalized to fulfill the orthogonality condition

$$\phi_m^t M \phi_m = I \tag{1}$$

where I is the unity matrix and the superscript t denotes matrix transpose. The normalization in Eq. (1) can be achieved by adjusting either the analytical mass matrix (e.g., Ref. 2) or the measured modes (e.g., Refs. 8 and 9) or the approach presented by the writers[10] that requires a smaller change for the measured modes than the method of Refs. 8 and 9.

Assume that $K(n \times n)$ is a symmetric stiffness matrix to be determined that satisfies the eigenvalue equation

$$K\phi_m = M\phi_m \Omega_m^2 \tag{2}$$

where $\Omega_m^2(m \times m)$ represents a diagonal matrix containing the measured system frequencies. If ϕ_h is the matrix of the remaining higher normalized modes of the system defined by M and K, the stiffness matrix will satisfy

$$K = M\phi_h\phi_h^t K\phi_h\phi_h^t M + M\phi_m\phi_m^t K\phi_m\phi_m^t M \tag{3}$$

From Eqs. (1) and (2), we have

$$\phi_m^t K\phi_m = \Omega_m^2 \tag{4}$$

Therefore,

$$K = M\phi_h\phi_h^t K\phi_h\phi_h^t M + M\phi_m\Omega_m^2\phi_m^t M \tag{5}$$

Note that although the unknown normalized higher modal matrix ϕ_h itself depends on the stiffness K, its product $\phi_h\phi_h^t$ can be determined from the given mass matrix and the product of the measured modes,

$$\phi_h\phi_h^t = M^{-1} - \phi_m\phi_m^t \tag{6}$$

Equation (5) is equivalent to Eq. (2). Any stiffness matrix that satisfies Eq. (5) will satisfy Eq. (2), and vice versa. Furthermore, any symmetric matrix used as an initial stiffness matrix in the right-hand side of Eq. (5) will yield an adjusted stiffness matrix that satisfies Eq. (2). Therefore, if K in the right-hand side of Eq. (5) is treated as a variable initial stiffness matrix, indicated by K_0, Eq. (5) becomes the general expression of the adjusted stiffness matrix,

$$K = M\phi_h\phi_h^t K_0\phi_h\phi_h^t M + M\phi_m\Omega_m^2\phi_m^t M \tag{7}$$

It is recognized that the substitution of the analytical stiffness matrix K_a for K_0 into Eq. (7) yields the solution of the Baruch/Berman method.

Generation of the Governing Equation

From Eq. (7) it is concluded that different methods for stiffness matrix adjustment based on Eq. (2) can be interpreted as different choices for the initial stiffness matrix K_0.

As in Ref. 5, we relate the initial stiffness K_0 to the analytical stiffness K_a by

$$K_0 = K_a \otimes \gamma \tag{8}$$

to consider the percentage change of the analytical stiffness elements. In Eq. (13), $\otimes$ represents an element-by-element multipli-

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cation operator, and γ_{ij} is the percentage variation of the stiffness coefficient K_{aij}. The error function to be minimized is defined as

$$\varepsilon = \|\hat{I} \otimes \gamma - \hat{I}\| = \sum_i \sum_j (\hat{I}_{ij}\gamma_{ij} - \hat{I}_{ij})^2 \tag{9}$$

with

$$\hat{I}_{ij} = \begin{cases} 1 & \text{if } K_{aij} \neq 0 \\ 0 & \text{if } K_{aij} = 0 \end{cases} \tag{10}$$

Usually, Eq. (2) is used as constraints in the optimization scheme.[5,6,11,12] Because Eq. (2) will be automatically satisfied by Eq. (7), we can use Eq. (7) to consider only the connectivities as constraints:

$$K_{ij} = \left[M\phi_h\phi_h^t(K_a \otimes \gamma)\phi_h\phi_h^t M + M\phi_m\Omega_m^2\phi_m^t M\right]_{ij} = 0 \qquad \{i, j \mid K_{aij} = 0\} \tag{11}$$

The number of independent constraints in Eq. (11) is equal to that of zero elements in the upper-side triangular part of the stiffness matrix. The Lagrange function is then defined as

$$L = \frac{1}{2}\|\hat{I} \otimes \gamma - \hat{I}\| + \sum_i \sum_j \lambda_{ij}\left[M\phi_h\phi_h^t(K_a \otimes \gamma)\phi_h\phi_h^t M + M\phi_m\Omega_m^2\phi_m^t M\right]_{ij} \tag{12}$$

where λ_{ij} are Lagrange multipliers and if $K_{aij} \neq 0$, then $\lambda_{ij} = 0$. Because of the symmetry of Eq. (11), $\lambda_{ji} = \lambda_{ij}$. To determine γ for minimal L, the partial derivatives of L with respect to each γ_{ij} are set equal to zero, which yields

$$(\hat{I} \otimes \gamma - \hat{I}) + K_a \otimes \left(\phi_h\phi_h^t M\lambda M\phi_h\phi_h^t\right) = 0 \tag{13}$$

Equation (13) is then pre-element-by-element multiplied by K_a and substituted into Eq. (11):

$$\left\{M\phi_h\phi_h^t\left[K_a \otimes K_a \otimes \left(\phi_h\phi_h^t M\lambda M\phi_h\phi_h^t\right)\right]\phi_h\phi_h^t M - D\right\}_{ij} = 0 \qquad \{i, j \mid K_{aij} = 0\} \tag{14}$$

with $D = M\phi_h\phi_h^t K_a\phi_h\phi_h^t M + M\phi_m\Omega_m^2\phi_m^t M$. Equation (14) is rewritten as

$$\sum_l \sum_m \lambda_{lm}\left(\sum_p \sum_q A_{ip}A_{lp}A_{mq}A_{jq}K_{apq}^2\right) = D_{ij} \qquad \{i, j \mid K_{aij} = 0; l, m \mid K_{alm} = 0\} \tag{15}$$

with $A = M\phi_h\phi_h^t = I - M\phi_m\phi_m^t$. By introducing the symmetry property, $\lambda_{ji} = \lambda_{ij}$, and rewriting Eq. (15) in matrix form, we obtain the governing equation for the Lagrange multipliers,

$$[C]\{\lambda\} = \{D\} \tag{16}$$

in which C is a square coefficient matrix of order $(s \times s)$ with s the number of zero elements in the upper-side triangular part of the original analytical stiffness. If $\{\lambda\}$ and $\{D\}$ are numbered in the same sequence, $[C]$ will be symmetrical. It can be shown that matrix C is nonnegative definite provided that the eigenequation (2) and the structural connectivity do not overdetermine the adjusted stiffness matrix.

Once λ is evaluated from Eq. (16), γ and K_0 can be obtained without any difficulty from Eqs. (13) and (8), respectively. Finally, the substitution of K_0 into Eq. (7) will produce the required adjusted stiffness matrix that preserves the structural connectivities and satisfies the eigenvalue equation (2).

Existence of Solution and Solution Algorithm

If K satisfies $K\phi_m = M\phi_m\Omega_m^2$, which provides $n \times m$ constraint equations, from $\phi_m^t M\phi_m = I$ it also satisfies $\phi_m^t K\phi_m = \Omega_m^2$. Because $K^t = K$, $\phi_m^t K\phi_m = \Omega_m^2$ has only $[m(m+1)]/2$ independent constraints. Therefore, the maximum number of independent constraints given by $K\phi_m = M\phi_m\Omega_m^2$ is[11]

$$q = n \times m - \left\{m \times m - \frac{m(m+1)}{2}\right\} = n \times m - \frac{m(m-1)}{2} \qquad m < n \tag{17}$$

If all of the elements of the adjusted stiffness matrix are to be determined, the unknowns will be $n(n+1)/2$ and greater than q given by Eq. (17) provided $m < n$, and an infinite number of adjusted stiffness matrices can be formulated, all satisfying $K\phi_m = M\phi_m\Omega_m^2$.

When some of the stiffness matrix elements are assumed to be zero due to the structural connectivities, the adjusted stiffness matrix may be over-, under-, or proper determined, depending on the number of measured modes used in the eigenvalue equation and the locations of the zero-valued stiffness elements.

We assume that an infinite number of solutions for the adjusted stiffness matrix exist, as is the usual case in practice. Correspondingly, Eq. (16) will have at least one solution for the Lagrange multipliers λ. Because the optimal procedure determines a unique solution for the stiffness percentage variation γ under the assumptions, any solution λ of Eq. (16) will yield the same optimal solution for γ from Eq. (13); this can be shown mathematically. This suggests that

Table 1 Adjusted stiffness elements

Element location	Corrupt element	Adjusted stiffness elements: Kabe[5,6] 1 mode (2 modes)	Kammer[7] 1 mode (2 modes)	Present 1 mode (2 modes)	Exact elements
1,1	15,750	15,543 (15,850)	15,540 (15,654)	15,511 (15,936)	13,750
1,2	−1,300	−1,299 (−1,325)	−1,299 (−1,318)	−1,301 (−1,325)	1,250
1,3	−1,300	−1,298 (−1,558)	−1,298 (−1,564)	−1,300 (−1,501)	−1,500
1,4	−12,000	−12,001 (−12,011)	−11,998 (−11,810)	−11,964 (−12,159)	−10,000
1,5	0	0 (0)	0 (0)	0 (0)	0
1,6	0	0 (0)	0 (0)	0 (0)	0
2,2	2,150	2,160 (2,253)	2,160 (2,252)	2,149 (2,253)	2,250
2,3	0	0 (0)	0 (0)	0 (0)	0
2,4	−850	−849 (−926)	−848 (−932)	−833 (−926)	−1000
2,5	0	0 (0)	0 (0)	0 (0)	0
2,6	0	0 (0)	0 (0)	0 (0)	0
3,3	2,150	2,163 (2,461)	2,161 (2,457)	2,157 (2,499)	2,500
3,4	−850	−849 (−899)	−848 (−888)	−842 (−998)	−1,000
3,5	0	0 (0)	0 (0)	0 (0)	0
3,6	0	0 (0)	0 (0)	0 (0)	0
4,4	22,900	23,310 (25,874)	23,326 (25,664)	23,244 (26,132)	24,000
4,5	−4,200	−4,208 (−5,000)	−4,211 (−5,000)	−4,205 (−5,000)	−5,000
4,6	−5,000	−5,037 (−7,000)	−5,056 (−7,000)	−5,036 (−7,000)	−7,000
5,5	5,100	5,066 (6,000)	5,071 (6,000)	5,064 (6,000)	6,000
5,6	0	0 (0)	0 (0)	0 (0)	0
6,6	5,900	5,787 (8,000)	5,809 (8,000)	5,786 (8,000)	8,000

that one can use the simple and effective QR decomposition to solve the governing equation (16). The rank of the matrix C can be determined numerically. The extra unknowns λ_{ij}, if any, can be simply set equal to zero and only the remaining λ_{ij} need to be solved to obtain a set of solutions for λ. This discussion and the solution algorithm are also applicable to the approaches of Kabe[5,6] and Caeser.[11,12]

Numerical Example

The same numerical example presented in Refs. 5 and 6 is used herein, so that a direct comparison can be made between the present method and the ones presented by Kabe[5,6] and Kammer.[7] The first and the first two measured modes are used in the adjustment procedure, and the results are shown in Table 1. As expected, all three methods produce similar results for the adjusted stiffness matrix.

Summary

Kabe[5,6] proposed a method that uses, in addition to the measured modes, the structural connectivities to optimally adjust the stiffness matrix. The weaknesses of the method are the large size of the governing equation and the complicated solution algorithm. In this Note, the Baruch/Berman solution[1–4] is generalized so that it enables the structural connectivities to be used as explicit constraints in the optimization procedure, and the dimension of the governing equation becomes identical to the number of zero elements in the upper-side triangular part of the stiffness matrix. Thus, the proposed method is suitable for the case of limited number of zero stiffness elements. The simple, efficient QR decomposition can be used to solve the governing system.

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